Fluctuating Air-Sea Interaction

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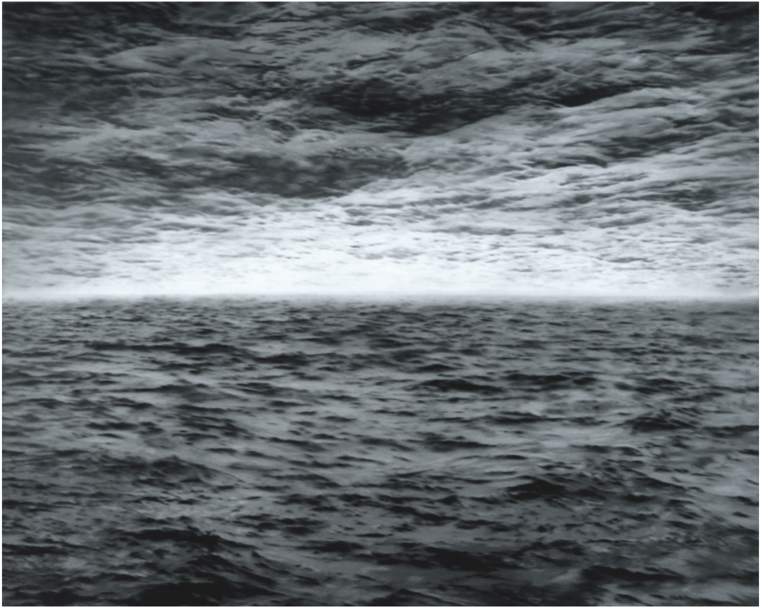
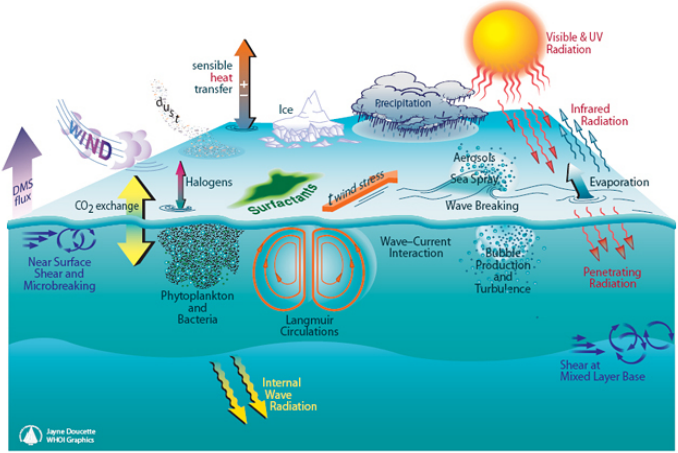
Outline

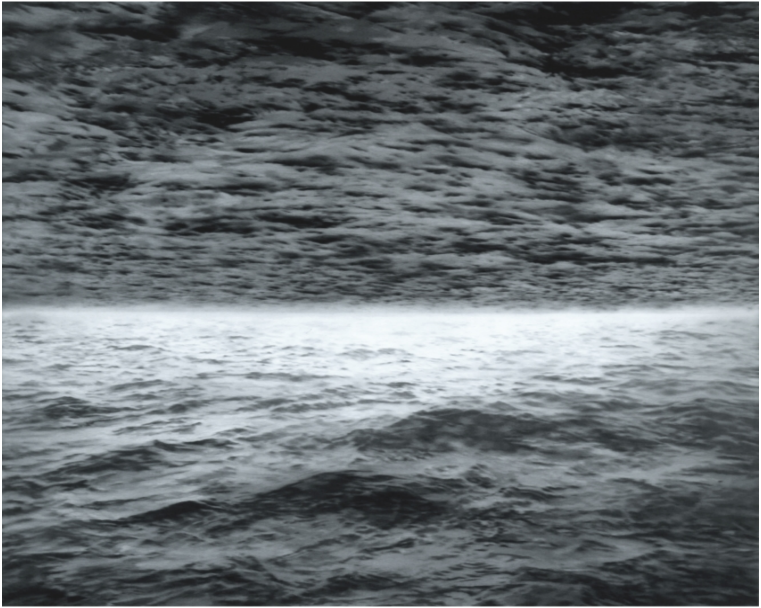
- Motivation : The Glass Transition
- Fluctuation Dissipation Relation (FDR)
- Fluctuation Dissipation Theorem (FDT)
- Fluctuation Relations (FR)
- Jarzynski equality and Crooks relation
- Conclusion / Perspectives



Seestück (G. Richter)

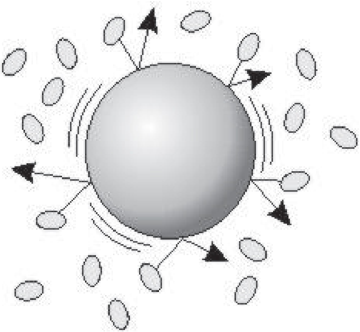
Air-Sea Interaction

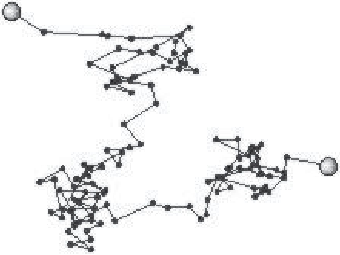




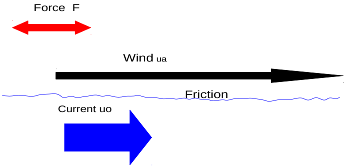
Seestück (G. Richter)

Brownian motion





Model



Parameters :

- mass ratio ocean/atmosphere: m
- friction coefficient (nonlinear): c_D

Einstein relation (1905)

- macroscopic: Stoke's law : $\gamma = \dfrac{6\pi\eta r}{m}$
- microscopic: random walk (1D): D = $\dfrac{\langle x^2 \rangle}{2t} = \dfrac{R}{\gamma^2}$
- equipartition : $\dfrac{k_B T}{m} = \langle u(t)^2 \rangle = \dfrac{R}{\gamma}$

$$D = \frac{k_B T}{\gamma m} = \frac{\mathrm{R}T}{N 6\pi\eta r}$$

Langevin Equation (1908)

$$m\partial_t u(t) = -m\gamma u(t) + F(t)$$

dissipation:	γ	macroscopic	systematic	constant
fluctuationn:	$F(t)$	microscopic	random	$\langle F(t)\rangle = 0$

$$\frac{m}{2}\partial_{tt}x^2 - mu^2 = -\frac{\gamma m}{2}\partial_t x^2 + xF$$

$$\frac{m}{2}\partial_{tt}\langle x^2\rangle - m\langle u^2\rangle = -\frac{m\gamma}{2}\partial_t\langle x^2\rangle + \cancel{\langle xF\rangle}$$

$$\frac{m}{2}\partial_t\langle\partial_t x^2\rangle + \frac{m\gamma}{2}\langle\partial_t x^2\rangle = k_B T$$

$$t \gg \frac{1}{\gamma} \quad \to \quad \langle\partial_t x^2\rangle = \frac{2k_B T}{m\gamma}$$

Langevin Equation, Itô calculus (1940)

$$u(0) = 0$$

$$u(t) = u(0)e^{-\gamma t} + e^{-\gamma t}\int_0^t F(t')e^{\gamma t'}dt'$$

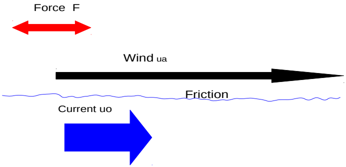
$$\langle u(t)^2 \rangle = e^{-2\gamma t}\int_0^t \int_0^t \langle F(t_1)F(t_2)\rangle e^{\gamma(t_1+t_2)} dt_2 dt_1$$

$$\langle F(t_1)F(t_2)\rangle = 2R\delta(t_2 - t_1)$$

Fluctuation dissipation relation:

$$\langle u(t)^2 \rangle = \frac{R}{\gamma}$$

Model

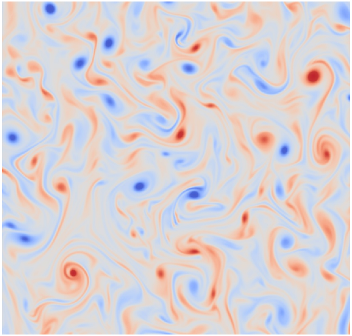


Parameters :

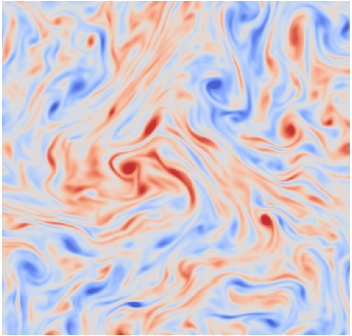
- mass ratio ocean/atmosphere: m
- friction coefficient (nonlinear): c_D

2D Turbulence

Atmos

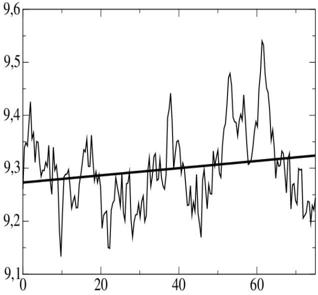


Ocean

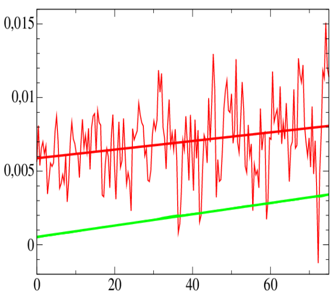


2D Turbulence

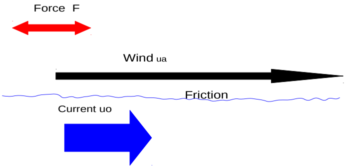
$\langle u_a^2 \rangle_A$



$\langle u_a u_o \rangle_A$, $\langle u_o^2 \rangle_A$



Model



$$
\begin{aligned}
\partial_t u_a &= -Sm(u_a - u_o) + F \\
\partial_t u_o &= -S \quad (u_o - u_a)
\end{aligned}
$$

Linear Local Model

$$
\begin{aligned}
\partial_t u_s &= -SMu_s + F \\
\partial_t u_t &= F
\end{aligned}
$$

$$
\begin{aligned}
u_s(t) &= \int_0^t e^{SM(t'-t)} F(t')dt' \\
u_t(t) &= \int_0^t F(t')dt'
\end{aligned}
$$

$$
\begin{aligned}
u_a(t) &= \frac{1}{M}(u_t + mu_s) = \frac{1}{M}\left(\int_0^t F(t')dt' + m\int_0^t e^{SM(t'-t)} F(t')dt'\right) \\
u_o(t) &= \frac{1}{M}(u_t - u_s) = \frac{1}{M}\left(\int_0^t F(t')dt' - \int_0^t e^{SM(t'-t)} F(t')dt'\right)
\end{aligned}
$$

Linear Local Model : 2nd order moments

$$\langle u_a^2\rangle_\Omega = \frac{R}{M^2}\left(2t + \frac{4m}{SM}(1-e^{-SMt}) + \frac{m^2}{SM}(1-e^{-2SMt})\right)$$

$$\langle u_o^2\rangle_\Omega = \frac{R}{M^2}\left(2t - \frac{4}{SM}(1-e^{-SMt}) + \frac{1}{SM}(1-e^{-2SMt})\right)$$

$$\langle u_a u_o\rangle_\Omega = \frac{R}{M^2}\left(2t + \frac{2(m-1)}{SM}(1-e^{-SMt}) - \frac{m}{SM}(1-e^{-2SMt})\right).$$

For $t \gg (SM)^{-1}$:

$$\langle (u_a - u_o)^2\rangle_\Omega = \frac{R}{SM}$$

$$\langle u_a^2 - u_o^2\rangle_\Omega = \frac{R(M+2)}{SM^2}$$

$$\langle u_a u_o - u_o^2\rangle_\Omega = \frac{R}{SM^2}$$

Fluctuation Dissipation Relation (FDR)

$$\frac{1}{2}\partial_t\langle u_o^2\rangle_\Omega = S\langle u_a u_o - u_o^2\rangle_\Omega = \frac{R(1-e^{-SMt})^2}{M^2}$$

For $t \gg (SM)^{-1}$:

$$\frac{R}{M^2} = \frac{SR}{M^2}\left(2t + \frac{m-2}{SM} - 2t + \frac{3}{SM}\right)$$

Quadratic Local Model

$$\begin{aligned} \partial_t \mathbf{u}_a &= - \quad \tilde{S} m |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \\ \partial_t \mathbf{u}_o &= \quad \tilde{S} \quad |\mathbf{u}_s| \mathbf{u}_s \end{aligned}$$

with $\mathbf{u}_s = \mathbf{u}_a - \mathbf{u}_o$, $\mathbf{u}_t = \mathbf{u}_a + m\mathbf{u}_o$.

$$\begin{aligned} \partial_t \mathbf{u}_s &= -\tilde{S} M |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} \\ \partial_t \mathbf{u}_t &= \mathbf{F} \end{aligned}$$

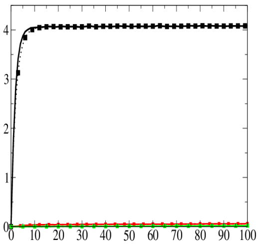
Linear Langevin eq. with eddy friction:

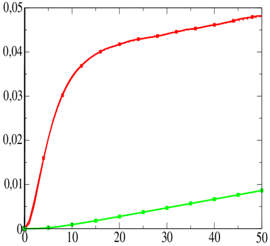
$$\frac{S_{\text{eddy}}}{\tilde{S}} = \frac{\langle (\mathbf{u}_s^2)^{3/2} \rangle}{\langle \mathbf{u}_s^2 \rangle^{3/2}} \langle (\mathbf{u}_s^2)^{1/2} \rangle = \left(\frac{\mu^2 2R}{\tilde{S} M} \right)^{1/3} .$$

$$\mu_{\text{Gaussian}} = \frac{\langle (\mathbf{u}_s^2)^{3/2} \rangle}{\langle \mathbf{u}_s^2 \rangle^{3/2}} = \frac{3\sqrt{\pi}}{4} \approx 1.3293404.$$

Lin. vs. Quadratic Langevin eq.

$$\langle u_a^2 \rangle_A, \langle u_o^2 \rangle_A, \langle u_a u_o \rangle_A$$





$$\mu = \frac{2}{3}\frac{\Gamma(2/3)}{\Gamma(4/3)} \approx 1.2449; (Gaussian) = \frac{3\sqrt{\pi}}{4} \approx 1.329$$

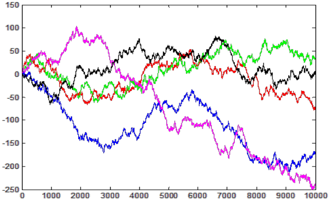
Stochastic differential equation:

Integrating many independent realisation:

$$\partial_t u = F(u, \omega) \quad \text{with,} \quad \omega \in \Omega$$

$\rightarrow$ measure moments :

$$\langle u^n \rangle_\Omega, \qquad \langle f(u) \rangle_\Omega$$



("Lagrangian approach")

Fokker-Planck equation:

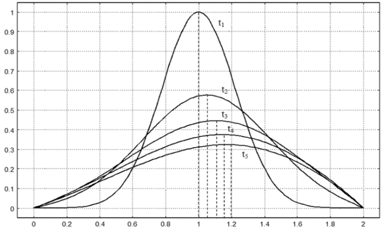
Obtain PDE for the time evolution of the pdf:

$$\partial_t P(u,t) = \partial_u \left(a(u)P(u) + \frac{1}{2}\partial_u \left[b(u)P(u)\right] \right)$$

$\rightarrow$ solve equation if possible and obtain moments by integration:

$$\langle u^n \rangle = \int u^n dP, \qquad \langle f(u) \rangle = \int f(u) dP$$

("Eulerian approach")



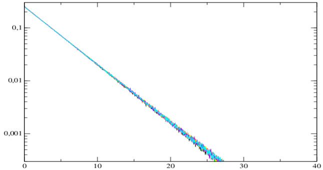
Linear model: SDE $\leftrightarrow$ Fokker-Planck equation:

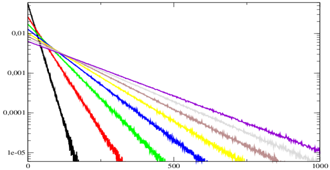
SDE:

$$
\begin{aligned}
\partial_t \mathbf{u}_s &= -SM\mathbf{u}_s + F \\
\partial_t \mathbf{u}_t &= F
\end{aligned}
$$

Fokker-Planck

$$
\begin{aligned}
\partial_t P_s &= \nabla_{uv} \cdot \left[SM\mathbf{u}_s P_s + \frac{1}{2} \nabla_{uv} P_s \right] \\
\partial_t P_t &= \frac{1}{2} \nabla_{uv} \cdot \nabla_{uv} P_t
\end{aligned}
$$





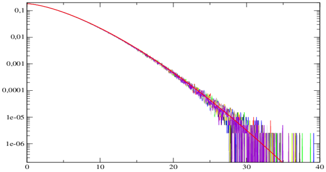
Non-linear model: SDE $\leftrightarrow$ Fokker-Planck equation:

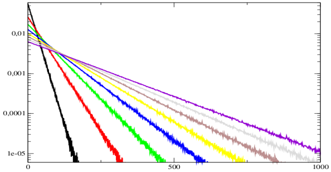
SDE:

$$
\begin{aligned}
\partial_t \mathbf{u}_s &= - \tilde{S} M |\mathbf{u}_s| \mathbf{u}_s + \mathbf{F} && (1) \\
\partial_t \mathbf{u}_t &= \mathbf{F} && (2)
\end{aligned}
$$

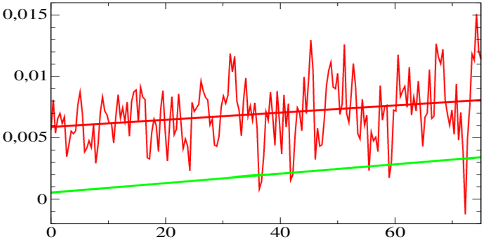
Fokker-Planck

$$
\begin{aligned}
\partial_t P_s &= \nabla_{uv} \cdot \left[\tilde{S} M \mathbf{u}_s u_s P_s + \frac{\nu}{2} \nabla_{uv} P_s \right] \\
\partial_t P_t &= \frac{\nu}{2} \nabla_{uv} \cdot \nabla_{uv} P_s
\end{aligned}
$$





FDR 2D : $\langle \mathbf{u}_o^2 \rangle$, $\langle u_a u_o \rangle$



$$\frac{1}{2}\partial_t \langle u_o^2 \rangle_A = S \langle u_a u_o - u_o^2 \rangle_A$$

$$\tilde{S}_{\mathrm{num}} = \frac{\partial_t \langle \mathbf{u}_o^2 \rangle}{2\mu_{\mathrm{Gauss}} \sqrt{\langle (\mathbf{u}_a - \mathbf{u}_o)^2 \rangle} \langle (\mathbf{u}_a \mathbf{u}_o - \mathbf{u}_o^2) \rangle}$$

$$\frac{\tilde{S}_{\mathrm{num}}}{\tilde{S}} = 0.9$$

(Wirth 2017, JPO)

Fluctuation Dissipation Theorem, Response Theory

Auto-correlation:

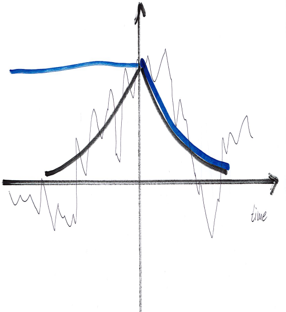
$$C(t, \Delta t) = \langle \mathbf{x}(t)\mathbf{x}^t(t + \Delta t)\rangle$$

Decay of a perturbation:

$$\langle \mathbf{x}(t + \Delta t)\rangle = \chi(t, \Delta t)\bar{\mathbf{x}}$$

The FDT:

$$C(t, \Delta t)C(t, 0)^{-1} = \chi(t, \Delta t).$$



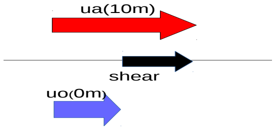
Fluctuation Dissipation Theorem (2)

The Fluctuation Dissipation Theorem is proofed for:

- linear models with white forcing.
- linear models with colored forcing, when the phase space is augmented by the forcing variable (otherwise dynamics at time t_0 is correlated to forcing at time $t > t_0$)

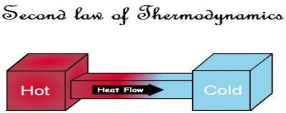
(Wirth 2021, NPG, paper of the month)

Power input (mechanical)



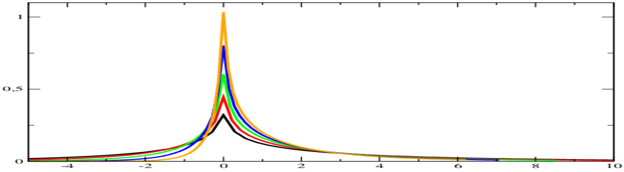
$$
\begin{aligned}
P &= \vec{\tau}\vec{u}_o \\
\tau &= C_D|\vec{u}_a - \vec{u}_o|(\vec{u}_a - \vec{u}_o) \\
\overline{Z}^{\tau} &= \frac{\int_t^{t+\tau} P(t')dt'}{\tau\langle P(t)\rangle}
\end{aligned}
$$

Fluctuation theorem



The pdf of time averages is considered.

$$\mathrm{Prob}(z_1 < \overline{Z}^{\tau} < z_2) = \int_{z_1}^{z_2} pdf_{\overline{Z}^{\tau}}(z)dz$$



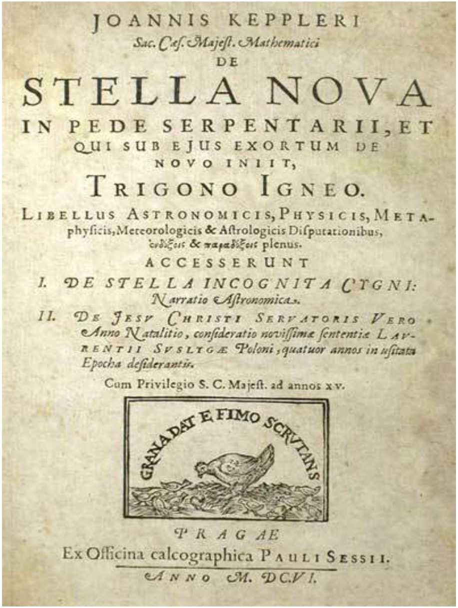
pdf is non Gaussian

Fluctuation theorem

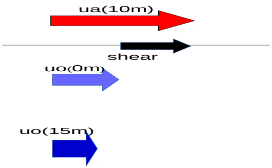
The symmetry function of the pdfs:

$$S_{\overline{Z}^{\tau}}(z) = \ln\left(\frac{pdf_{\overline{Z}^{\tau}}(z)}{pdf_{\overline{Z}^{\tau}}(-z)}\right) = \sigma\tau z,$$

(Wirth 2019, NPG, paper of the month)


JOANNIS KEPPLERI
Sac. Cæs. Majest. Mathematici
DE
STELLA NOVA
IN PEDE SERPENTARII, ET
QUI SUB EJUS EXORTUM DE
NOVO INIIT,
TRIGONO IGNEO.
LIBELLUS ASTRONOMICIS, PHYSICIS, META-
physicis, Meteorologicis & Astrologicis Disputationibus,
ἐνδόξοις & παραδόξοις plenus.
ACCESSERUNT
I. DE STELLA INCOGNITA CYGNI:
Narratio Astronomica.
II. DE JESU CHRISTI SERVATORIS VERO
Anno Natalitio, consideratio novissimæ sententiæ LAU-
RENTII SUSLYGÆ Poloni, quatuor annos in usitata
Epocha desiderantis.
Cum Privilegio S. C. Majest. ad annos XV.
GRANA DAT E FIMO SCRUTANS
PRAGAE
Ex Officina calcographica PAULI SESSII.
ANNO M. DCVI.

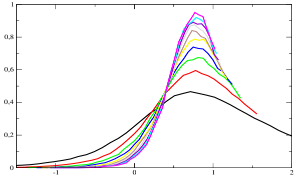
Power input (mechanical)



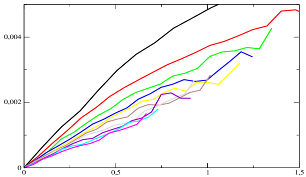
$$
\begin{aligned}
P &= \vec{\tau} \cdot \vec{u}_o(15m) \\
\vec{\tau} &= C_D |\vec{u}_a(10m) - \vec{u}_o(0m)| (\vec{u}_a(10m) - \vec{u}_o(0m))
\end{aligned}
$$

Fluctuation theorem

$(20^o - 30^o N, 20^o - 30^o W)$, res 0.5^o (sub-trop. gyre)
1993–2017, res 6h



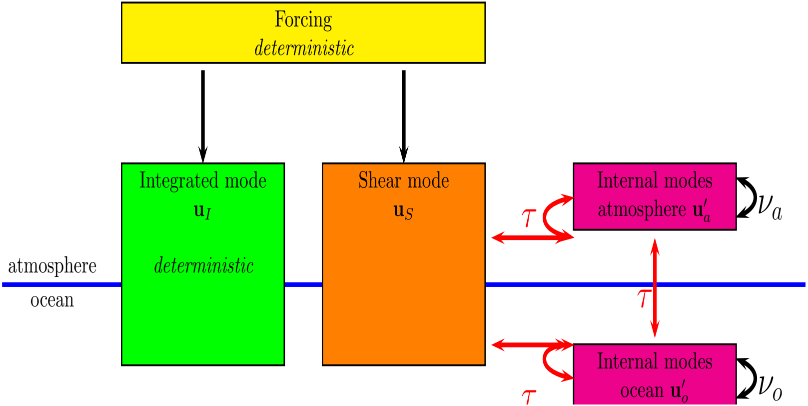
p(z)



S(z)/τ

- Pdf non Gaussian
- With increasing averaging time negative events for the power-input to the ocean occure less often.
- The symmetry function is linear with z and scales $\propto \tau$.

Jarzynski equality and Crooks relation



Jarzynski equality and Crooks relation

- Jarzynski equality:

$$\langle e^{-\beta w} \rangle_f = e^{-\beta \Delta G} \tag{3}$$

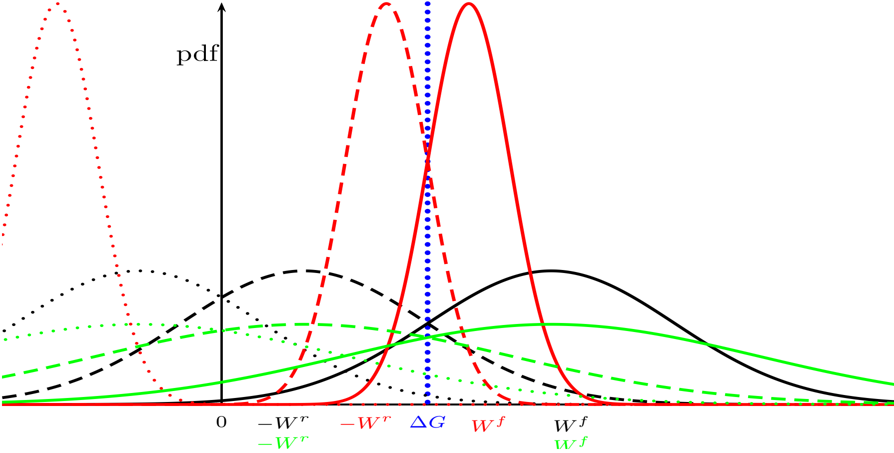
- Crooks relation:

$$\frac{\mathrm{pdf}^f(w)}{\mathrm{pdf}^r(-w)} = \exp(\beta_D[w - \Delta G]) = \exp(-\beta_D q). \tag{4}$$

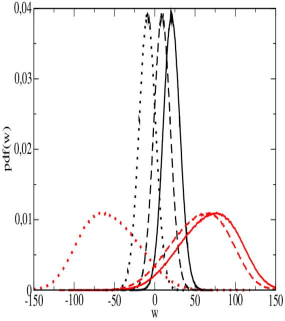
- Integral fluctuation theorem:

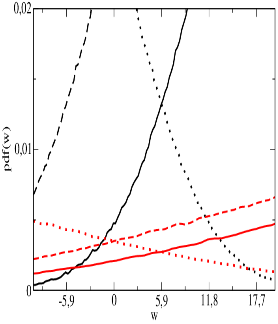
$$1 = \left\langle \frac{\mathrm{pdf}^r(-w)}{\mathrm{pdf}^f(w)} \right\rangle_f = \langle \exp(\beta_D q) \rangle_f. \tag{5}$$

PDF

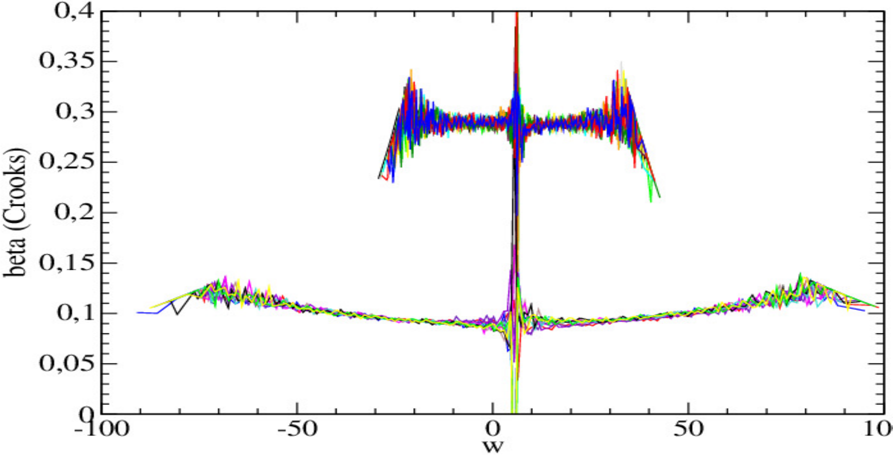


PDF (non-linear)





β_D



Conclusions

* The ocean subject to atmospheric forcing obeys a fluctuation dissipation relation.
* Local models (linear and quadratic) can be solved analytically (also with coloured noise).
* Some of the results from local models can be transposed to fully 2D turbulence models.
* FDT, FT, Jarzynski equality and Crooks relation are explored.

Perspectives

- Dissipation of non-resolved dynamics is included in models (atmosphere, ocean climate, ...) but not the fluctuations. However, fluctuation-dissipation-relations hold at all levels of the dynamics.
- Consider truely non equilibrium processes (beyond: spin-up, spin-down)
- Glassy states $\rightarrow$ Look at co-organization between ocean and atmosphere dynamics
- Use modern tools of nonequilibrium stat. mech. in climate science
- Applies whenever two systems with different characteristic scales interact

Data, Data, Data