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Spontaneous stochasticity or a tale on the "true" butterfly effect

based on joint work with Alexei Mailybaev (IMPA, Rio de Janeiro), Simon Thalabard (INPHYNI, Nice), Nicolas Valade (Inria, Sophia Antipolis)

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Lorenz's butterfly effect

Sensitive dependence of dynamical systems
upon initial perturbations/errors *(E.N. Lorenz 1963, 1969)*

Lyapunov exponent & Chaos: $\dot{\boldsymbol{X}}_t = \boldsymbol{F}(\boldsymbol{X}_t) \qquad \boldsymbol{X}'_t = \boldsymbol{X}_t + \delta\boldsymbol{X}_t$

Tangent/linearised system: $\delta\dot{\boldsymbol{X}}_t = \nabla_{\boldsymbol{X}}\boldsymbol{F}(\boldsymbol{X}_t)\,\delta\boldsymbol{X}_t \quad \Rightarrow |\delta\boldsymbol{X}_t| \simeq |\delta\boldsymbol{X}_0|\,\mathrm{e}^{\lambda\,t}$

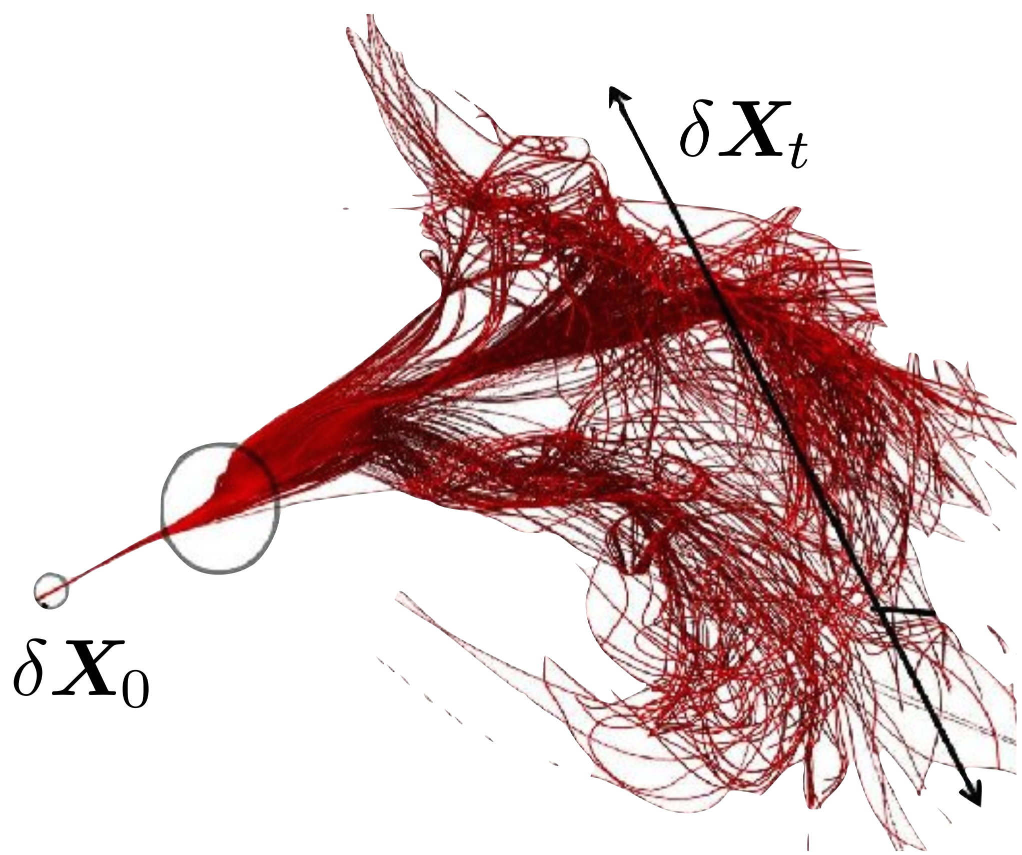
Lorenz actually distinguished between **two kinds of behaviours**:

- Error can be made arbitrarily small by reducing sufficiently the initial discrepancy.
- The two replica of the systems reach diverging states, no matter how small they differ initially.

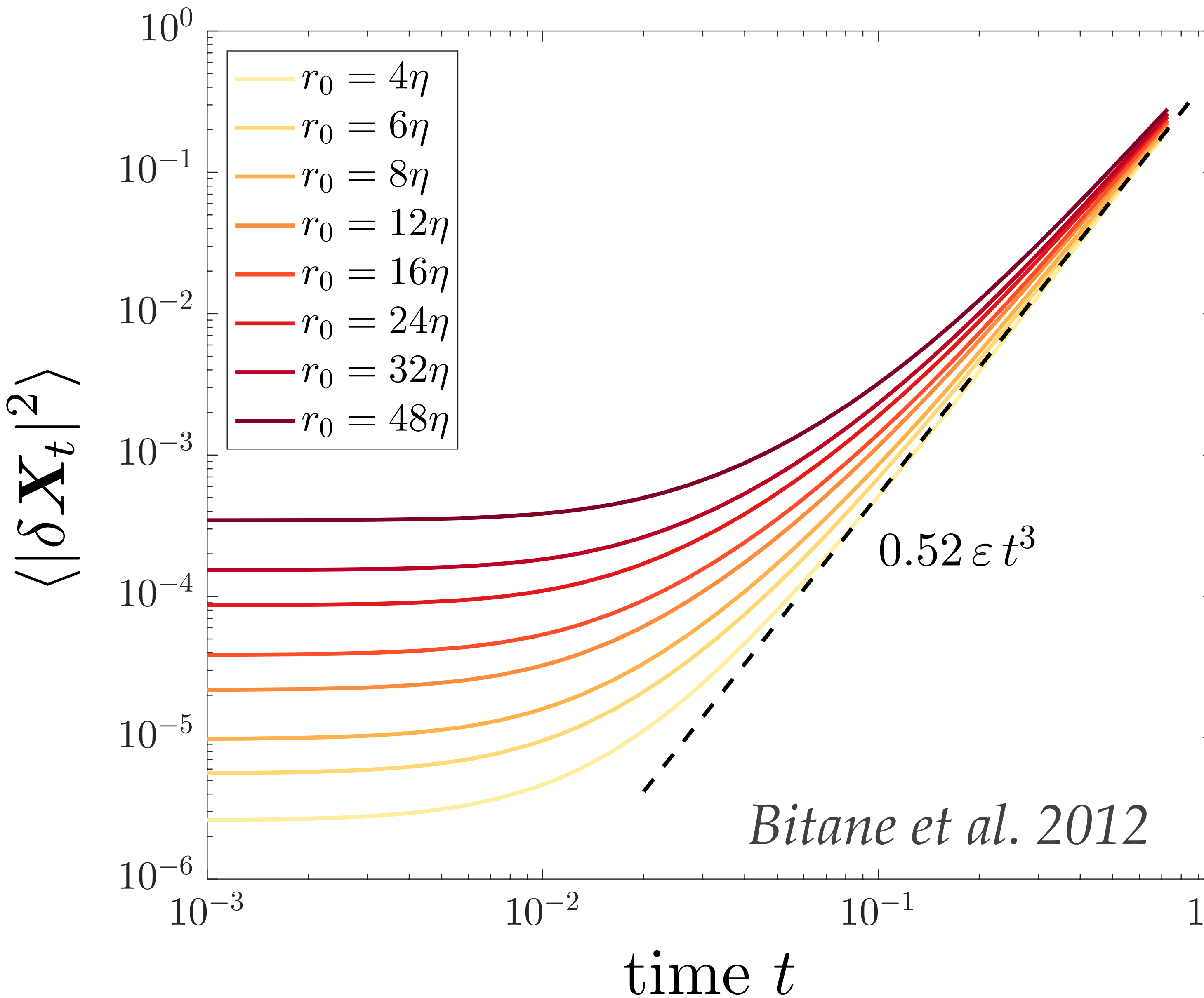
Explosive separation of tracers in 3D turbulent flow $\dot{\boldsymbol{X}}_t = \boldsymbol{u}(\boldsymbol{X}_t, t)$
(L.F. Richardson 1926)

Dissipative anomaly: $|\boldsymbol{u}(\boldsymbol{x}', t) - \boldsymbol{u}(\boldsymbol{x}, t)| \sim |\boldsymbol{x}' - \boldsymbol{x}|^{1/3}$ (non-Lipschitz)

Super-diffusive separation: $\delta \dot{X}_t \sim \delta X_t^{1/3} \;\Rightarrow\; \delta X_t \propto t^{3/2}$



from Scatamacchia et al. 2013



Bitane et al. 2012

Heuristics of Lagrangian dispersion

1D proxy: $\delta \dot{X}_t \approx \dfrac{\alpha\,\delta X_t}{(\delta_\nu + |\delta X_t|)^{1-h}} + \sqrt{4\nu}\,\eta_t$

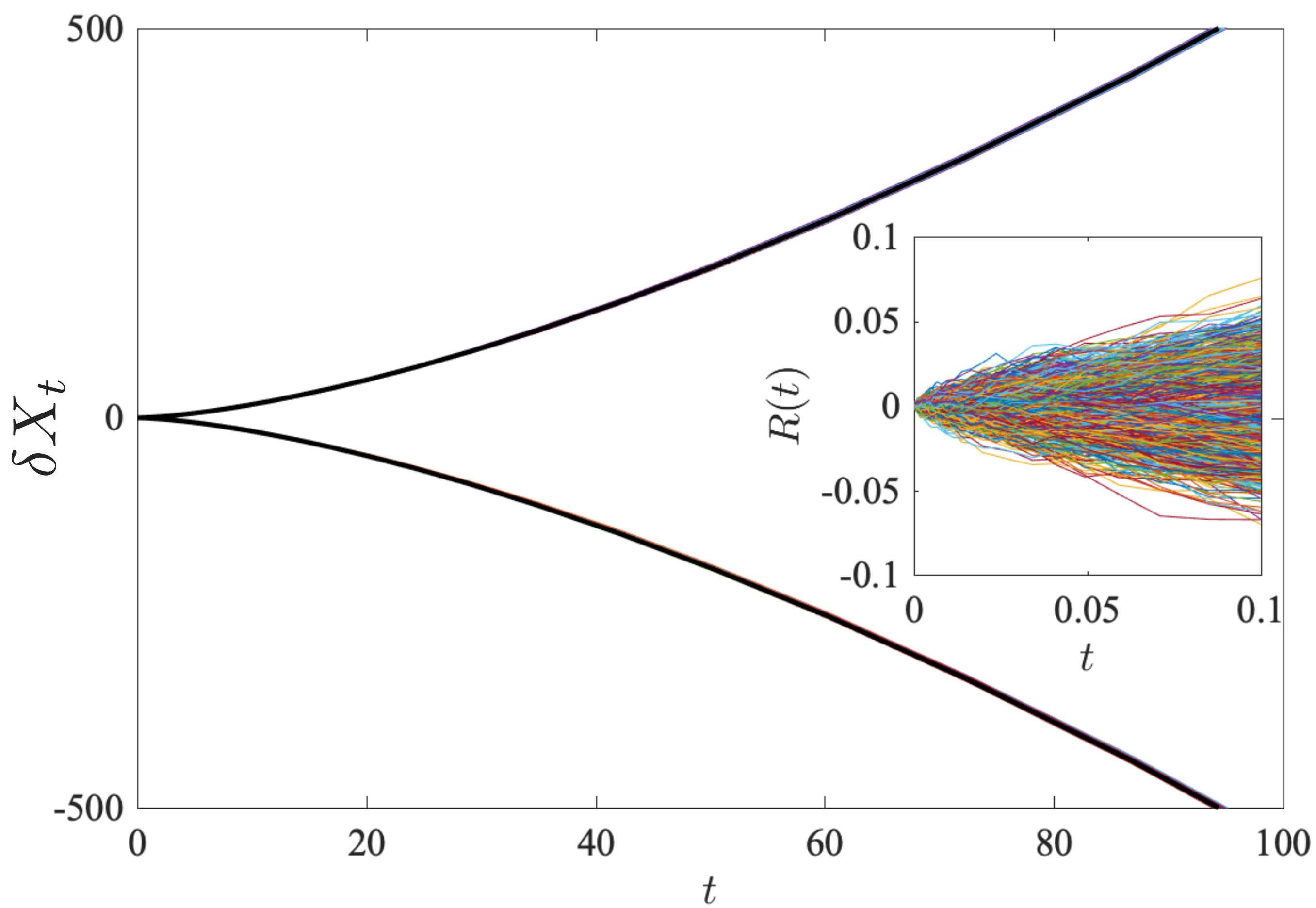
$\delta_\nu \approx$ Kolmogorov's scale

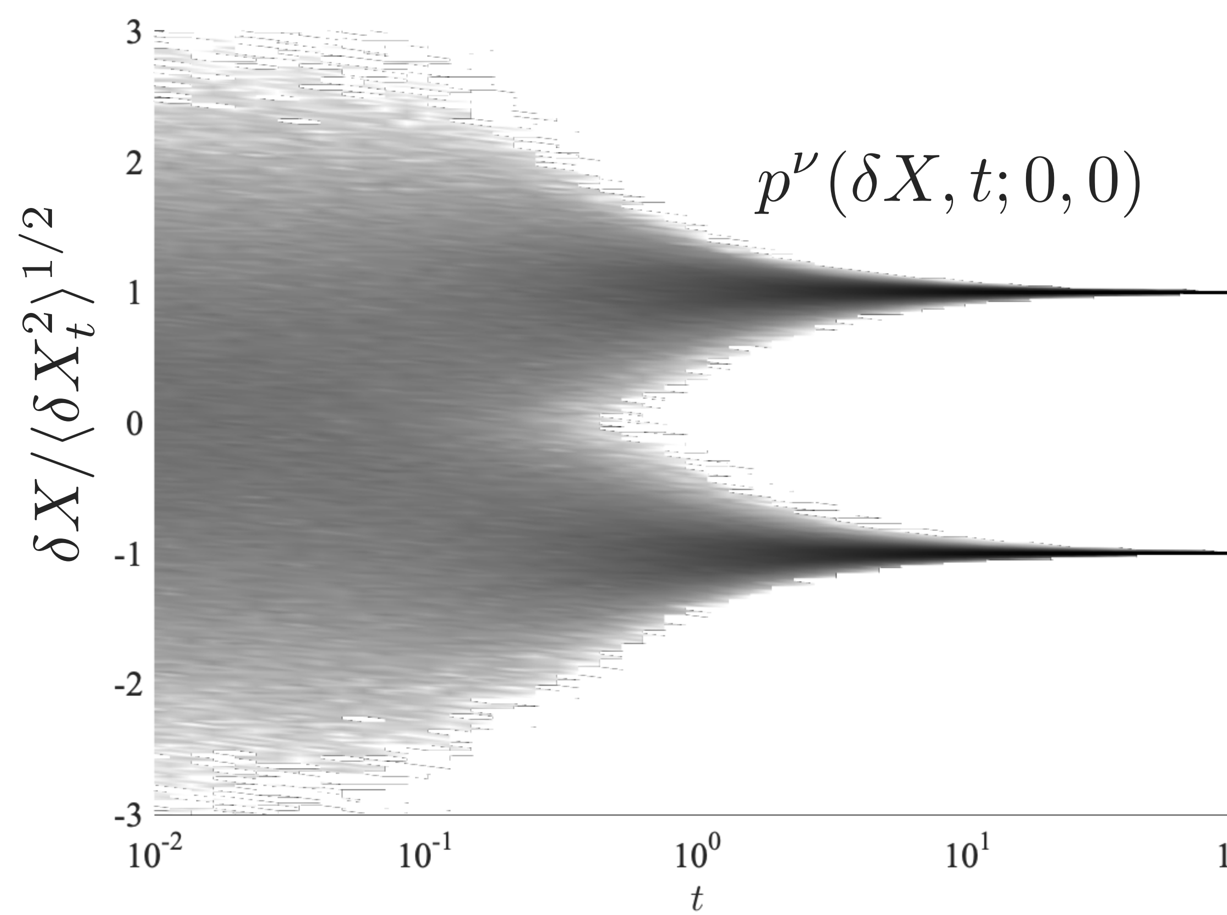
$\delta_\nu \xrightarrow[\nu\to 0]{} 0$

$\propto |\delta X_t|^h$ for $\delta X_t \gg \delta_\nu$ (non-Lipschitz)

$\propto \delta X_t$ for $\delta X_t \ll \delta_\nu$ (differentiable)

Mean-squared dispersion: $\langle \delta X_t^2 \rangle \simeq D\, t^{\frac{2}{1-h}}$





$$\lim_{\nu\to 0} p^\nu(\delta X, t; 0, 0) = \frac{1}{2}\delta\left(\delta X - D\, t^{\frac{1}{1-h}}\right) + \frac{1}{2}\delta\left(\delta X + D\, t^{\frac{1}{1-h}}\right) \neq \delta(\delta X)$$

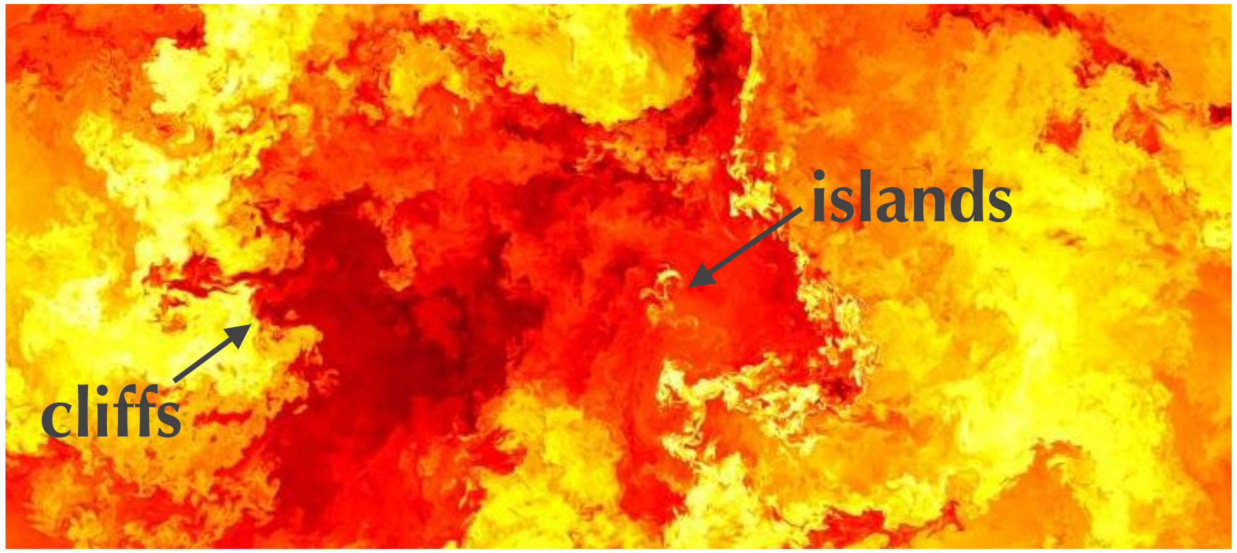
Turbulent Lagrangian flow

The Lagrangian flow is **ill-defined** in the limit $\nu \to 0$ (i.e. $Re \to \infty$)

⇒ largely explains **anomalous dissipation** and **intermittency** of advected passive scalars

Bernard et al. 1998; Pumir et al. 2000; Falkovich et al. 2001; Sawford 2001; Eyink 2008

Distance travelled by tracers



Impacts on the Eulerian velocity field? $$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho_f} \nabla p$$

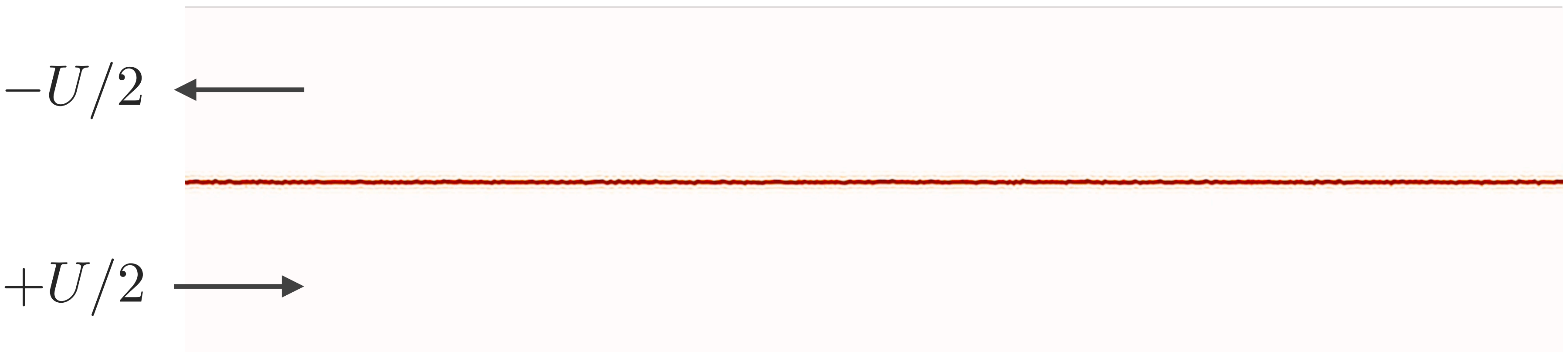
the velocity field transports itself!

- Does an explosive separation between the fluid element trajectories implies that velocity is not only singular but also ill-defined?
- Could this constitute a universal mechanism preventing uniqueness of singular solutions to the inviscid dynamics?

Eulerian spontaneous stochasticity

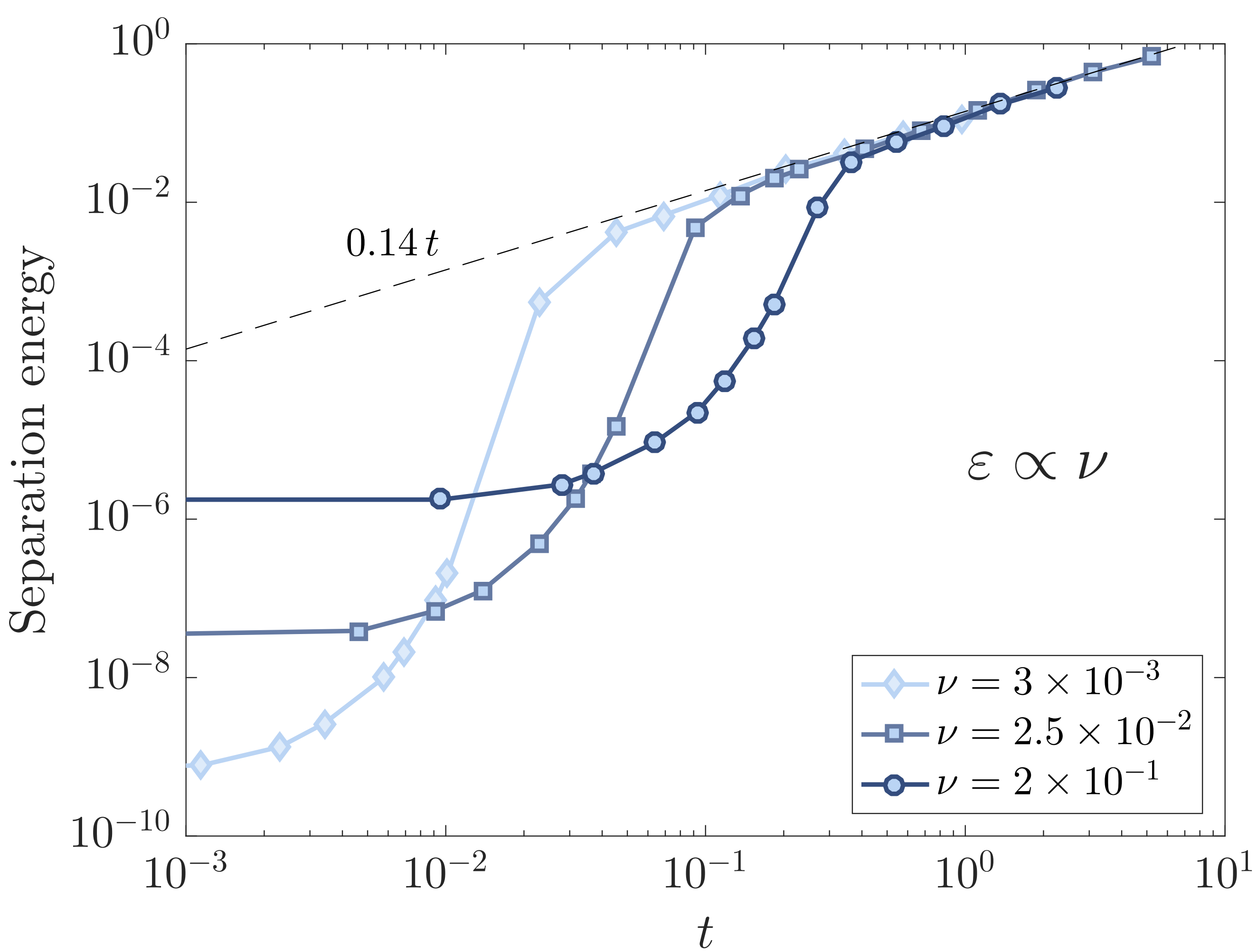
Example: 2D Kelvin–Helmholtz vortex sheet *Thalabard, Bec, Mailybaev 2020*

singular velocity field + various regularisations (ν) + small noise (ε)



The mixing layer reaches a finite size in a finite time, even when ε, $\nu \to 0$

Explosive separation of **Eulerian** trajectories



- **Discontinuity** w.r.t. initial data
- **Universal**, intrinsically stochastic nature of the inviscid dynamics

Predictability is infinitely less than in any chaotic system

What about geophysical flows & climate models?

(Rotunno & Snyder 2008, Palmer et al. 2014)

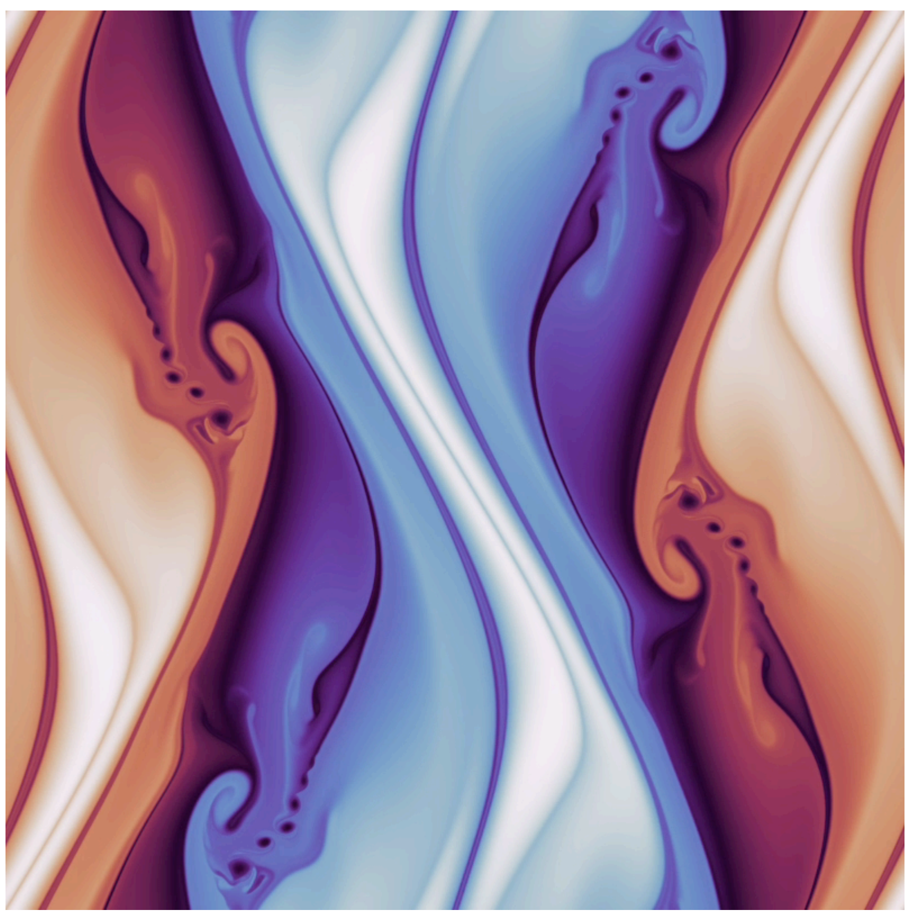
Surface quasi-geostrophic turbulence

Active transport of temperature, 2D analog of 3D hydro turbulence

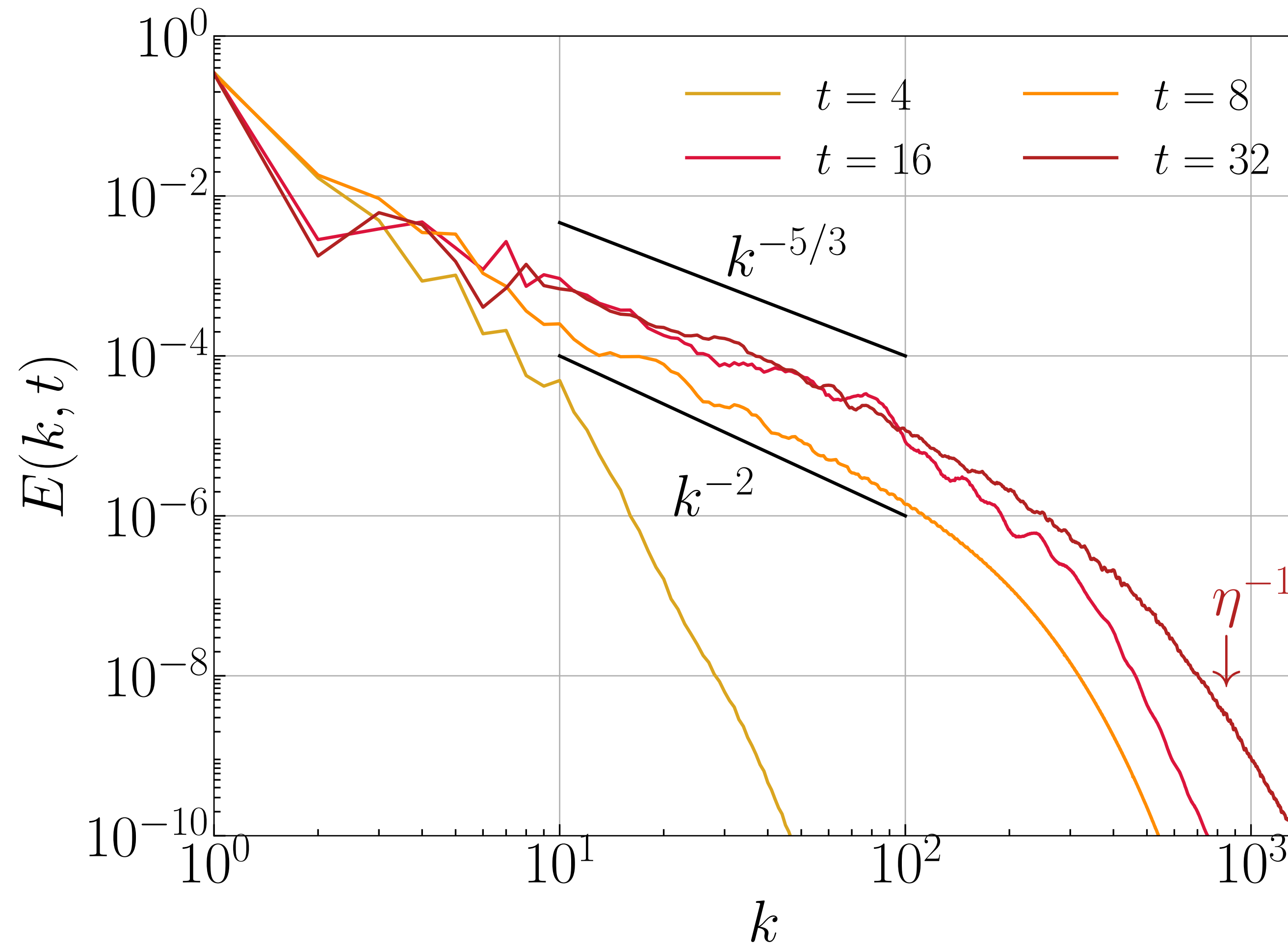
SQG equation: $\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \nu \Delta \theta + \phi$ where $\boldsymbol{u} = \nabla^{\perp} \Psi$ with $|\Delta|^{1/2} \Psi = \theta$ *Lapeyre 2017*

Turbulence from analytical initial data: $\theta_0(\boldsymbol{x}) = \cos x_2 - \sin x_1 \sin x_2$

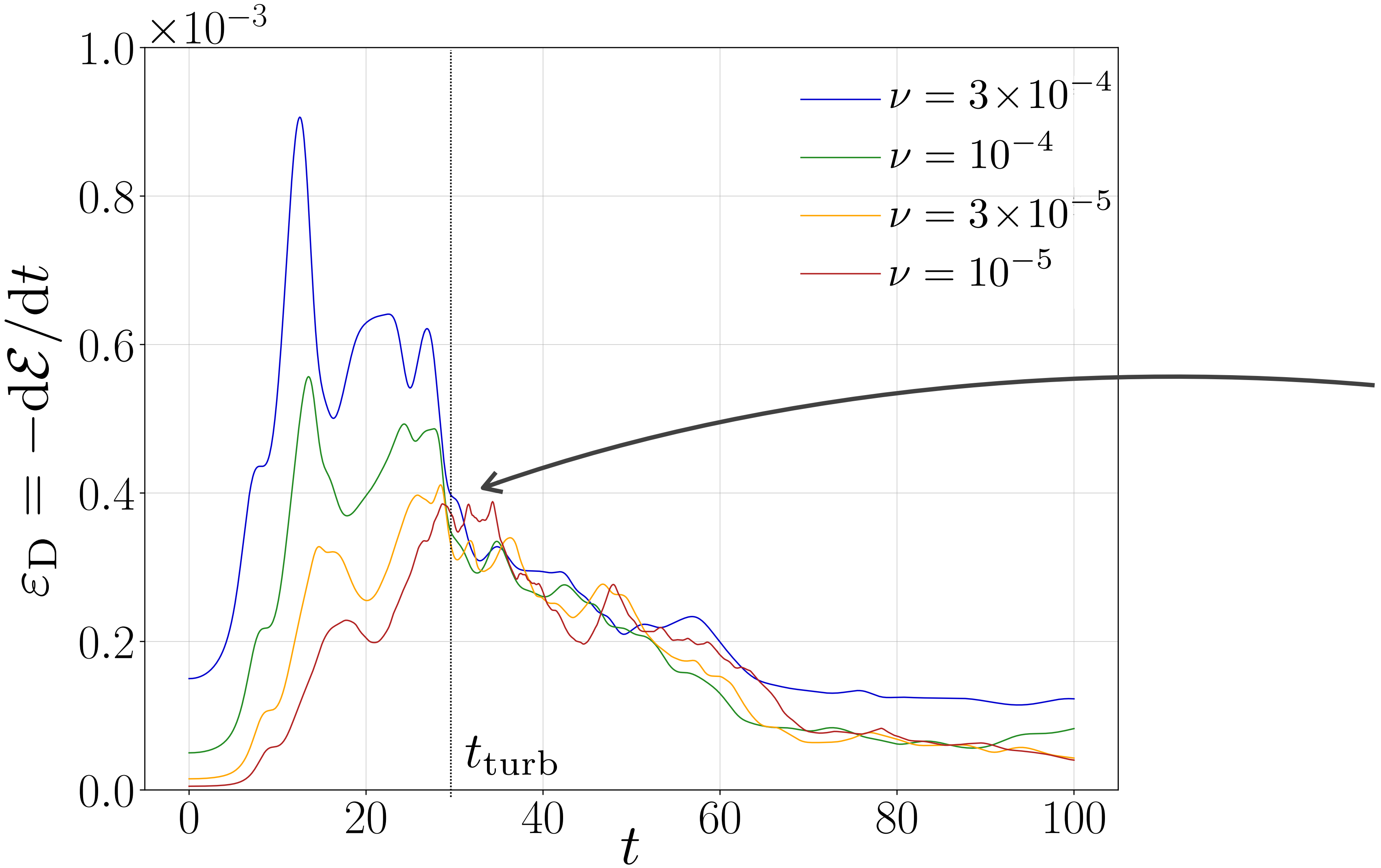
Constantin, Majda, Tabak 1994

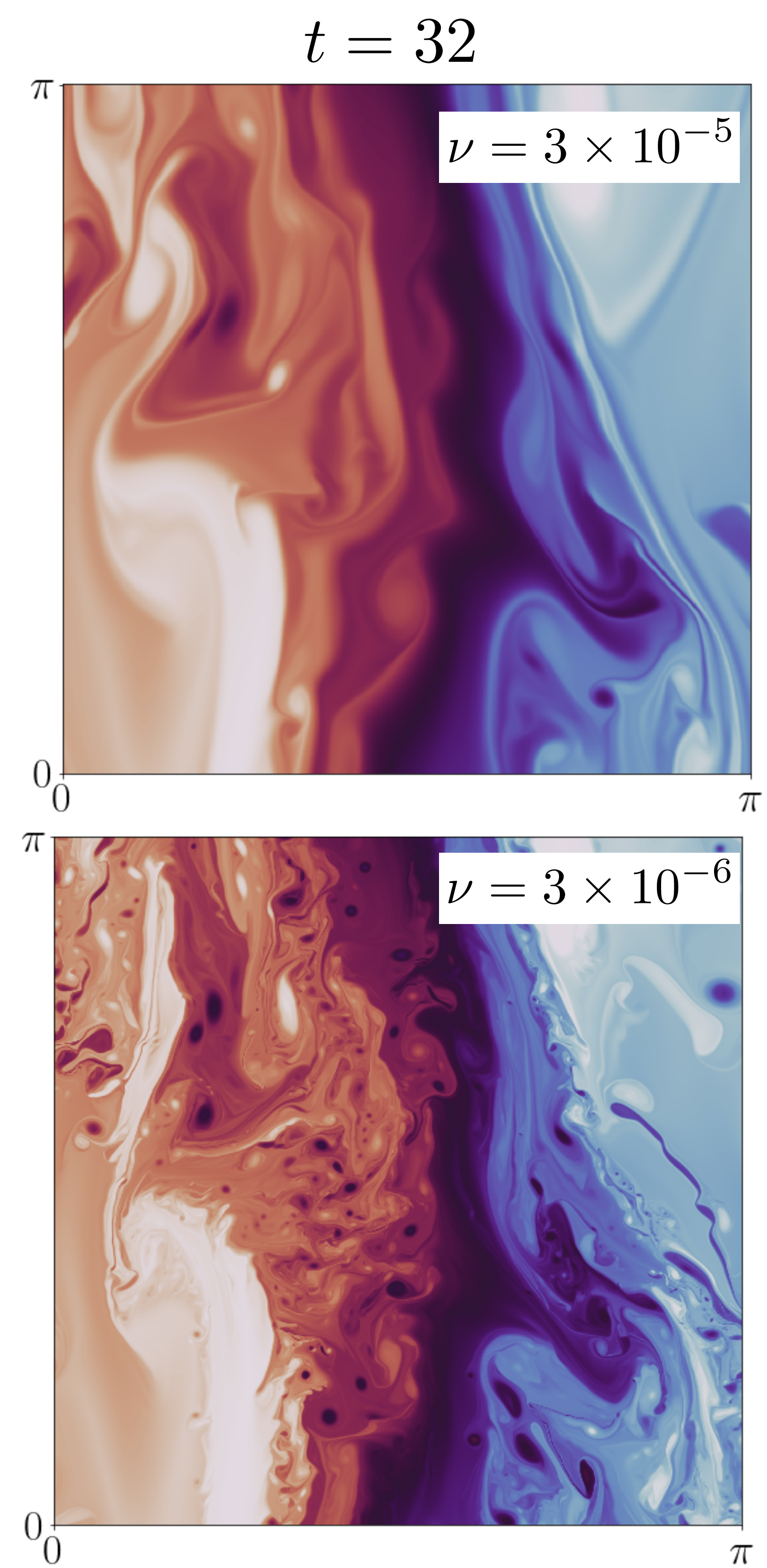


Valade, Thalabard, Bec (2023)



Dissipative anomaly





Solutions exhibit a turbulent regime for $t > t_{\text{turb}}$ regardless of any potential blowup for $\nu = 0$

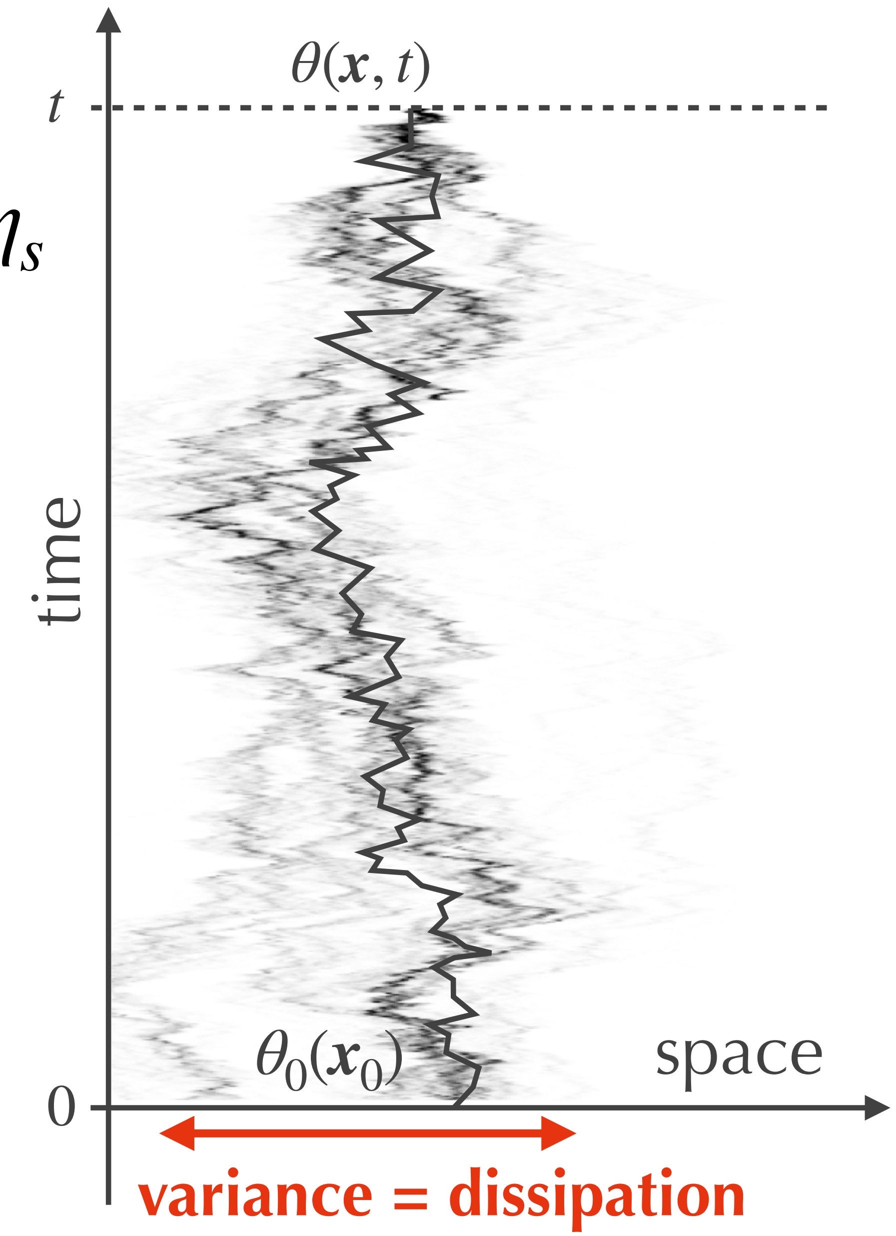
Dissipation & Lagrangian flow

- Feynman–Kac: $\theta(\boldsymbol{x}, t) = \mathbb{E}^{\nu}[\theta_0(\boldsymbol{X}_0)]$
  with average over tracers: $\dot{\boldsymbol{X}}_s = \boldsymbol{u}(\boldsymbol{X}_s, s) + \sqrt{2\nu}\,\eta_s$
  with final condition $\boldsymbol{X}_t = \boldsymbol{x}$

- "Fluctuation-dissipation" relation:

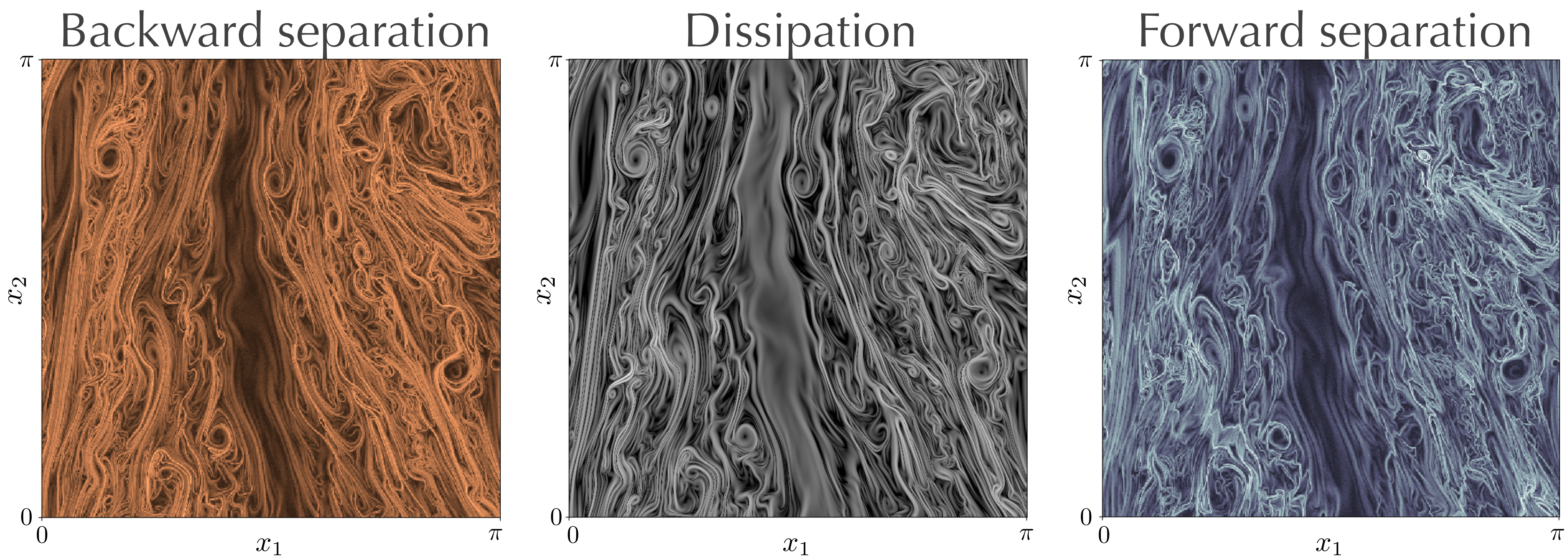
$$\mathscr{E}(0) - \mathscr{E}(t) = \frac{1}{8\pi^2}\iint [\theta(\boldsymbol{x}, t) - \theta_0(\boldsymbol{x}_0)]^2 p^{\nu}(\boldsymbol{x}_0, 0\,|\,\boldsymbol{x}, t)\, \mathrm{d}^2x\, \mathrm{d}^2x_0$$

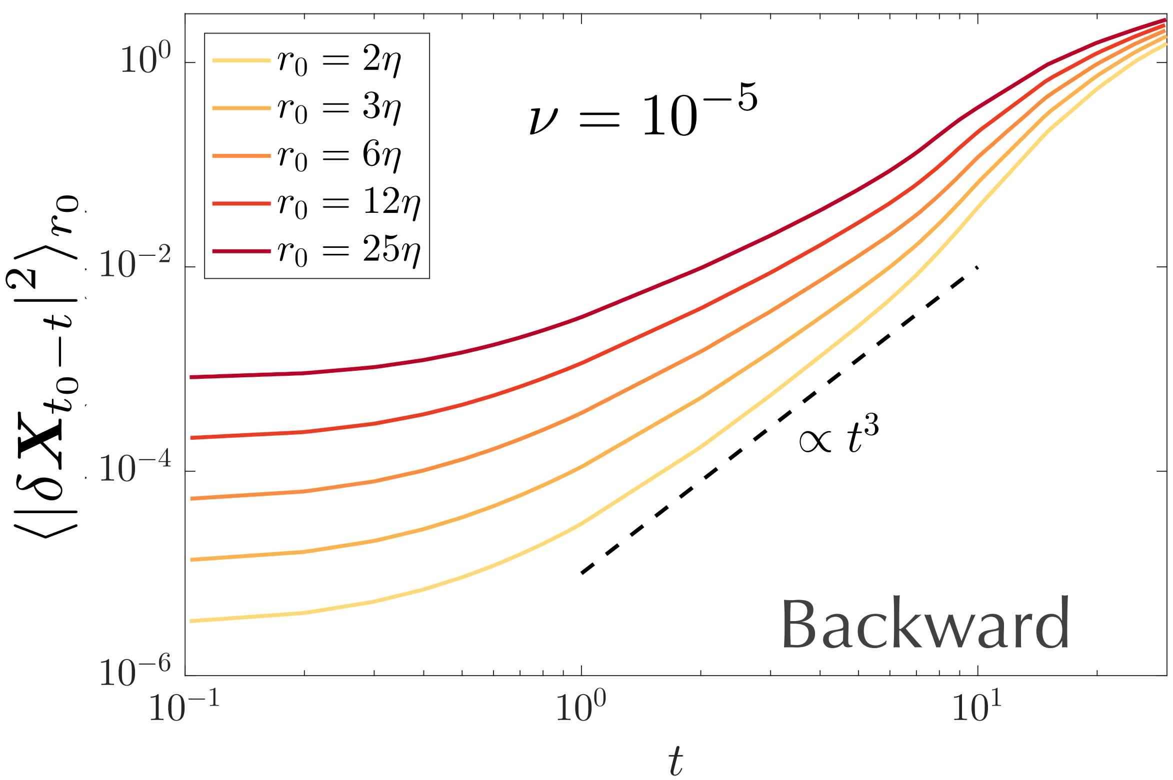
$p^{\nu}(\boldsymbol{x}_0, t_0\,|\,\boldsymbol{x}, t)$ transition pdf of tracers dynamics

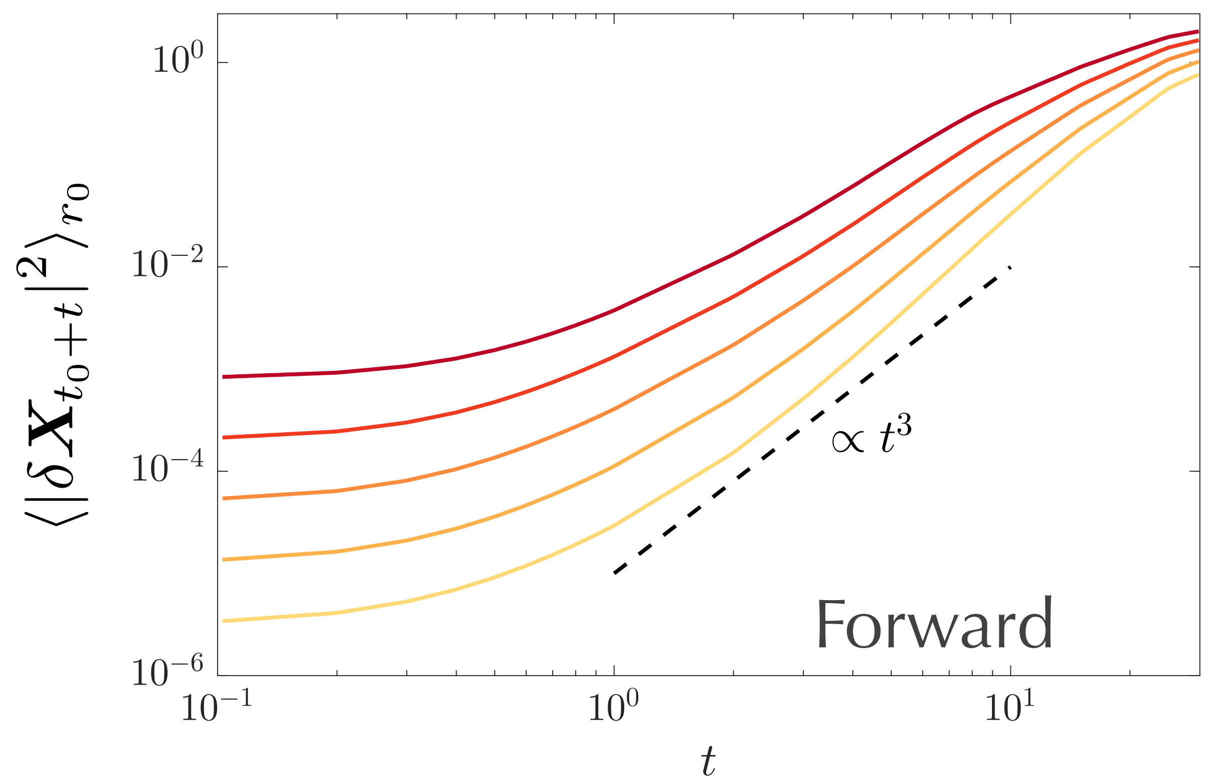


- Dissipative anomaly: **breakdown of the backward Lagrangian flow**
  i.e. $\lim_{\nu \to 0} p^{\nu}(\boldsymbol{x}_0, 0\,|\,\boldsymbol{x}, t) \neq \delta(\boldsymbol{X}_0 - \boldsymbol{x}_0)$

Lagrangian spontaneous stochasticity



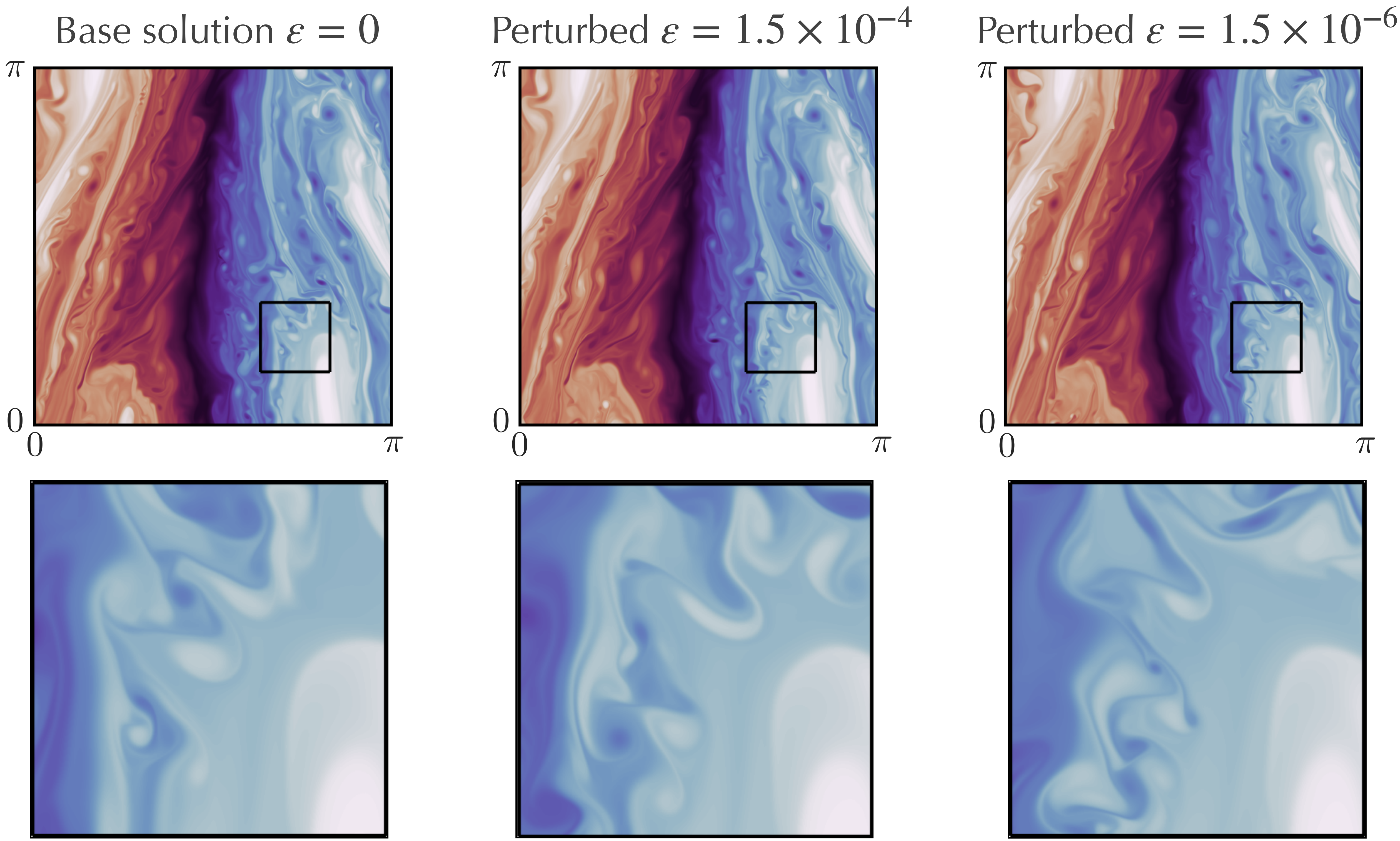




Relative dispersion keeps a noticeable dependence upon initial separation

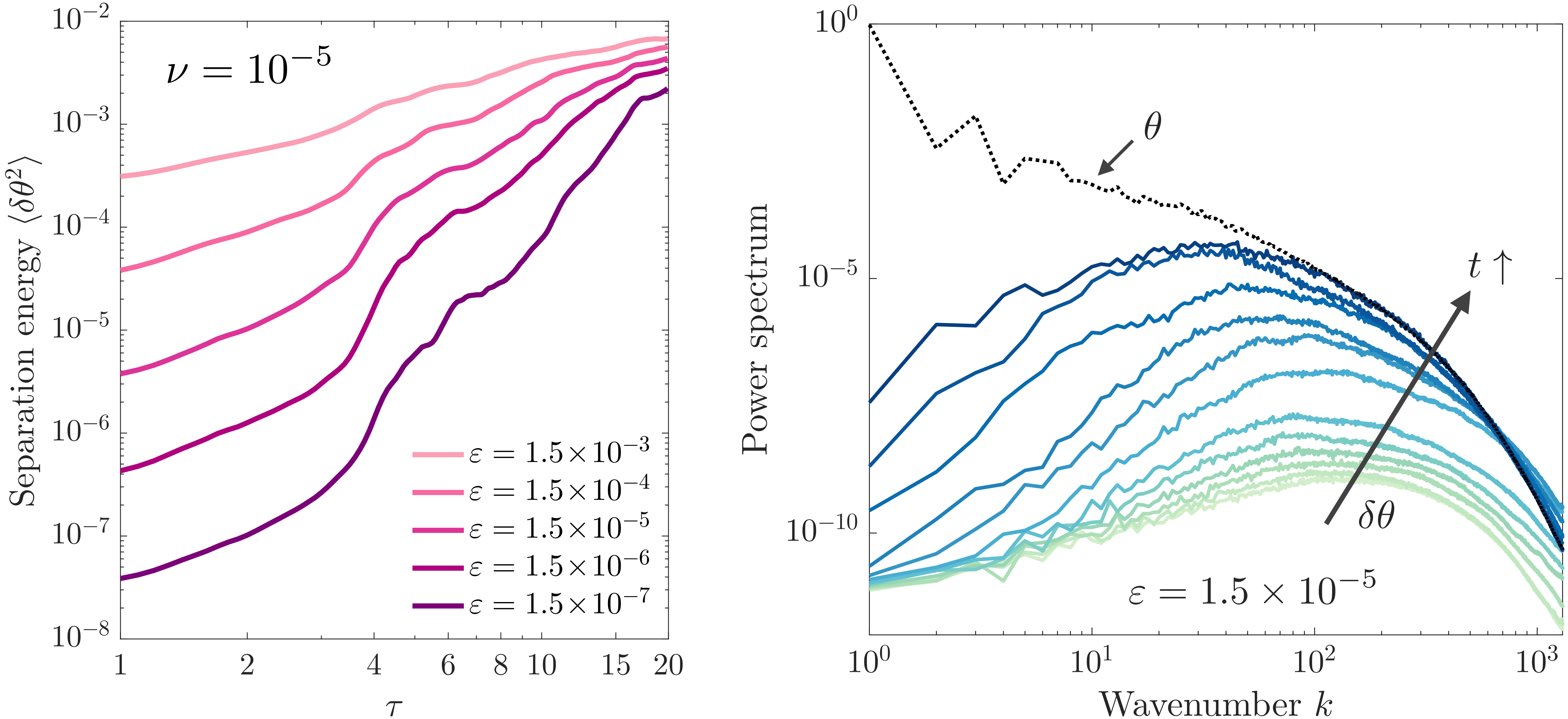
Eulerian spontaneous stochasticity

Perturbation at time $t = 30$ with spatial noise of energy ε



Eulerian spontaneous stochasticity

At $t = t_0 > t_{\text{turb}}$, small-amplitude perturbation: $\theta' = \theta + \delta\theta$, $|\delta\theta| = \varepsilon$



- Dependence upon perturbation size ε is again persisting at large t
- Inverse cascade of errors, but growth is not self-similar

Conclusions

- Spontaneous stochasticity provides a modern framework to address Lorenz's ideas on unpredictability from a probabilistic viewpoint

  Strong connections with the breakdown of the Lagrangian flow certainly exist.

- This intrinsic randomness is associated with the singular behaviour of dissipative inviscid solutions

  It may be necessary to **relax the notion of velocity field** to describe flows at infinite Reynolds numbers *(DiPerna & Majda 1987, Brenier 1989)*.

  Connection with **non-uniqueness** of weak dissipative solutions? (*De Lellis and Székelyhidi 2010, Brenier et al. 2011; Buckmaster & Vicol 2019)*

- The spontaneous stochasticity of SQG flow appears to be "tempered"

  Dependence on initial condition may not be differentiable (chaos), nor discontinuous (spontaneous stochasticity), but rather **singular** in between.

- Move beyond toy models: These ideas need to be extended to more realistic & complex models of relevance to geophysical flows